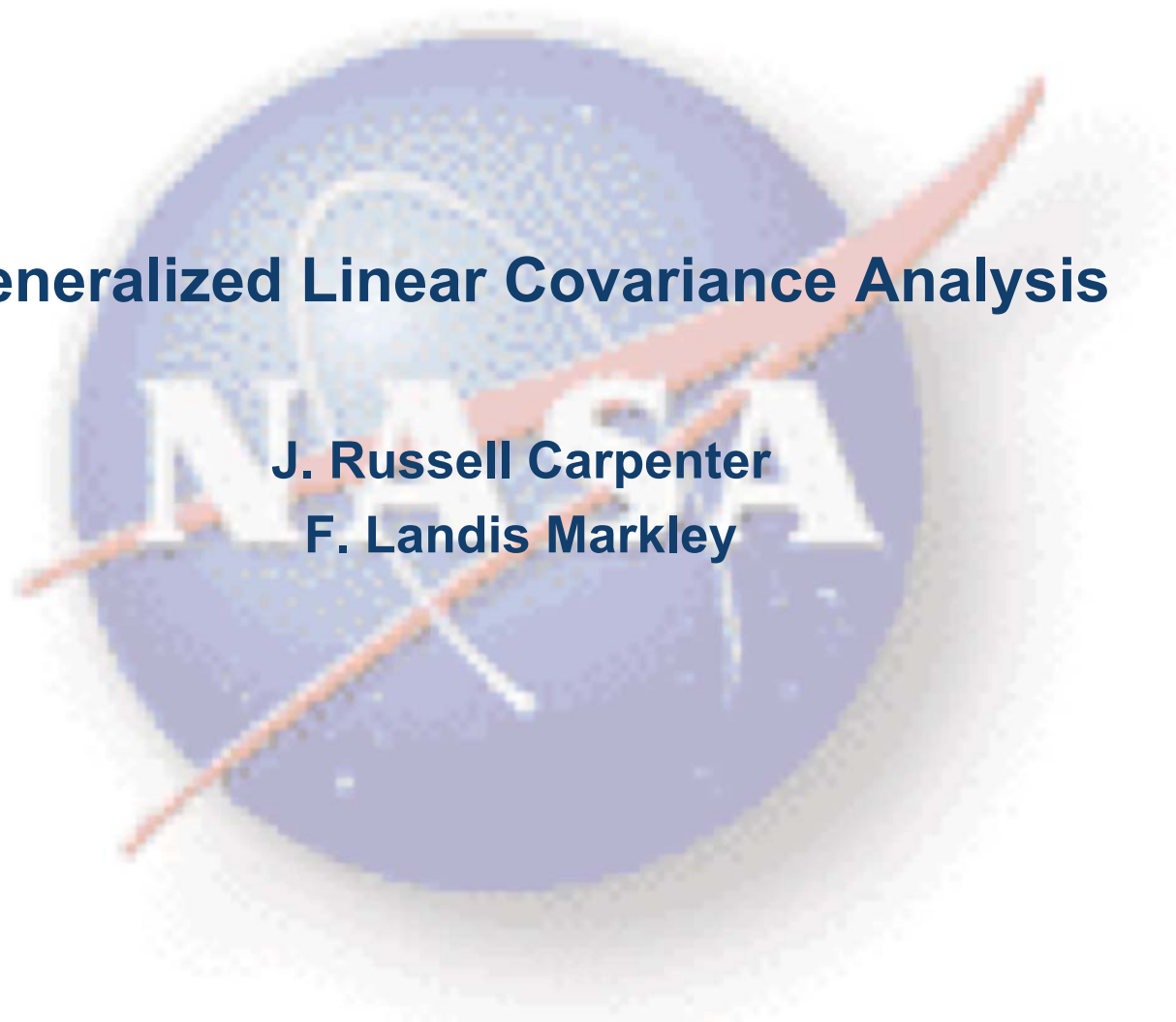
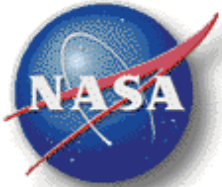


# **Generalized Linear Covariance Analysis**

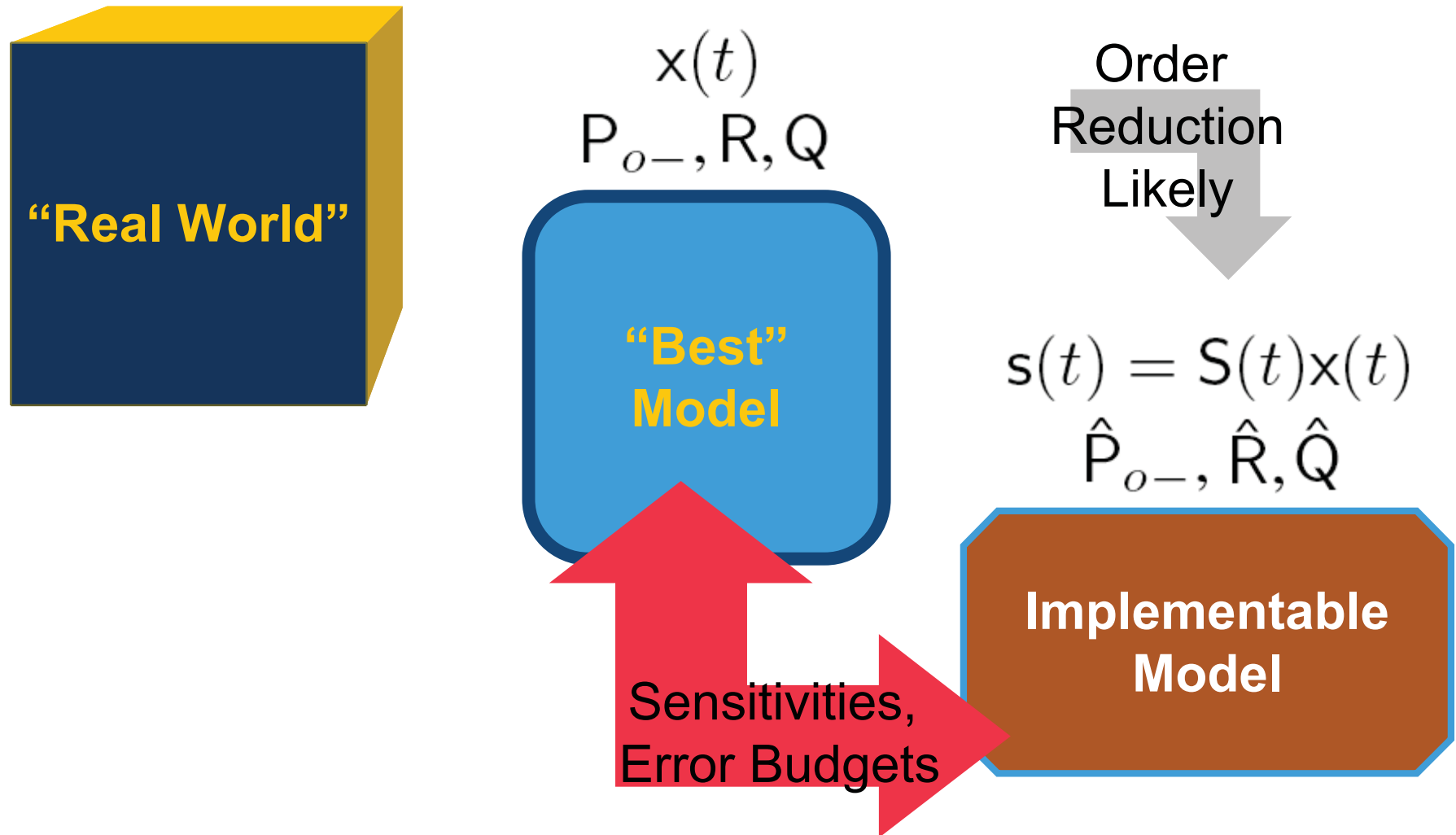
**J. Russell Carpenter**  
**F. Landis Markley**

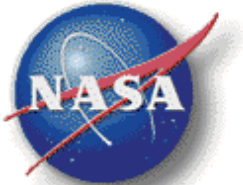




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## Motivation



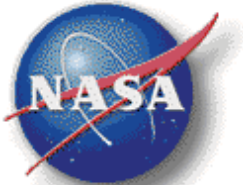


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## Some Background

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- Jazwinski (1970) - “solve-for/consider” state framework; assumes sequential filter
- Gelb (1974) - “true/filter” state framework (“true state” is “difference” between filter state and truth state); assumes sequential filter
- Maybeck (1979) - variation on Gelb; uses true (linear) state, rather than deviation from filter state
- Markley, et al. (1988) - uses solve-for consider framework, explicitly models contributions from a priori, measurement noise, and process noise for batch and sequential
- Tapley, et al. (2004) - uses solve-for/consider framework for batch and sequential

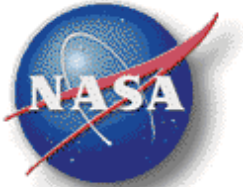


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## Present Work

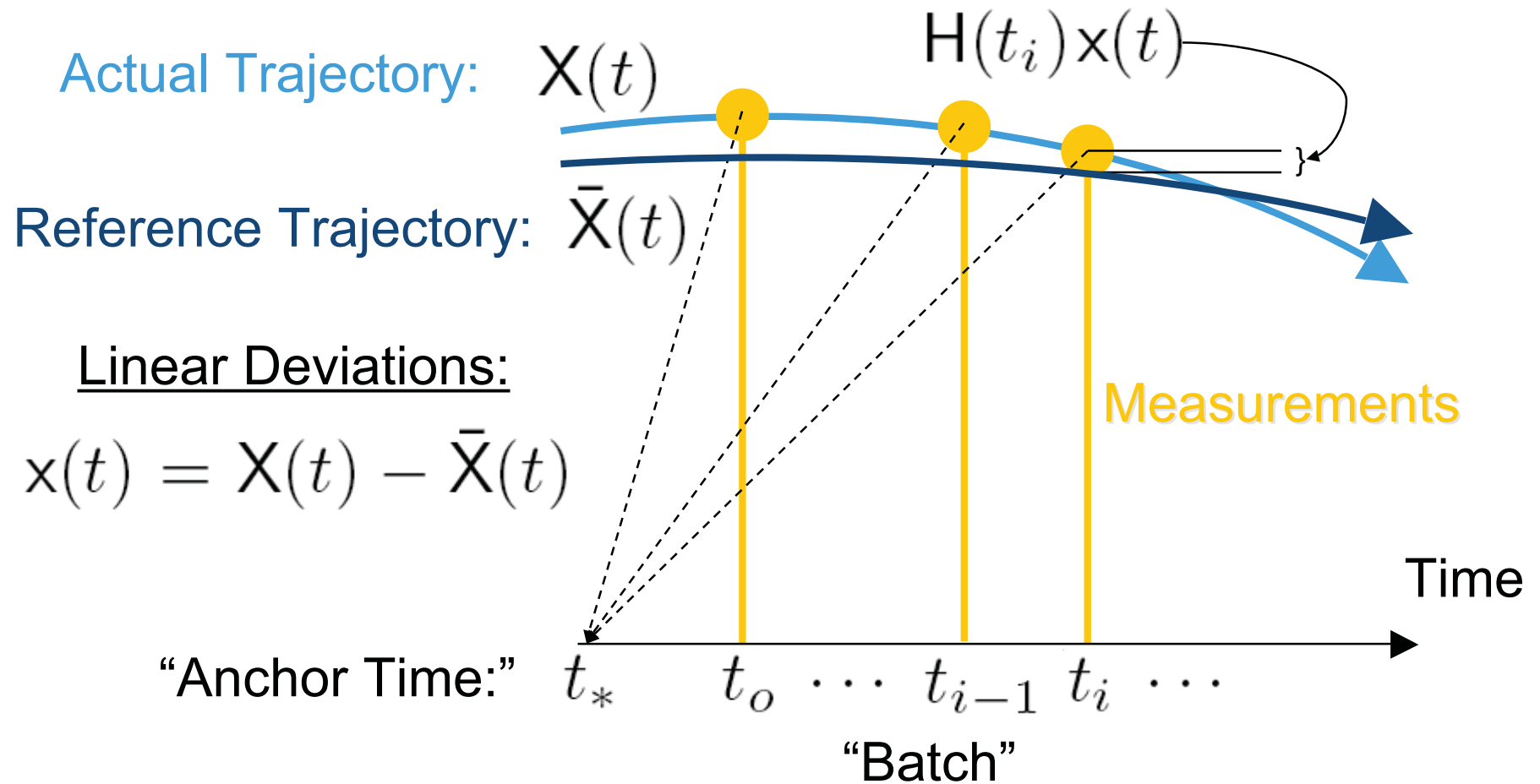
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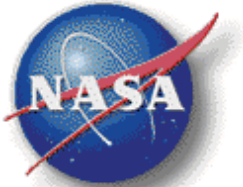
- **Update to Markley et al. ('88,'89)**
- **Explicitly addresses sensitivity of solve-fors with respect to considers in context of '88 paper's covariance partitions**
- **Explicitly addresses “postdiction” in the batch framework**
- **Applies method to integrated orbit/attitude problem**
- **Includes some new ways to examine output**



## Context ...

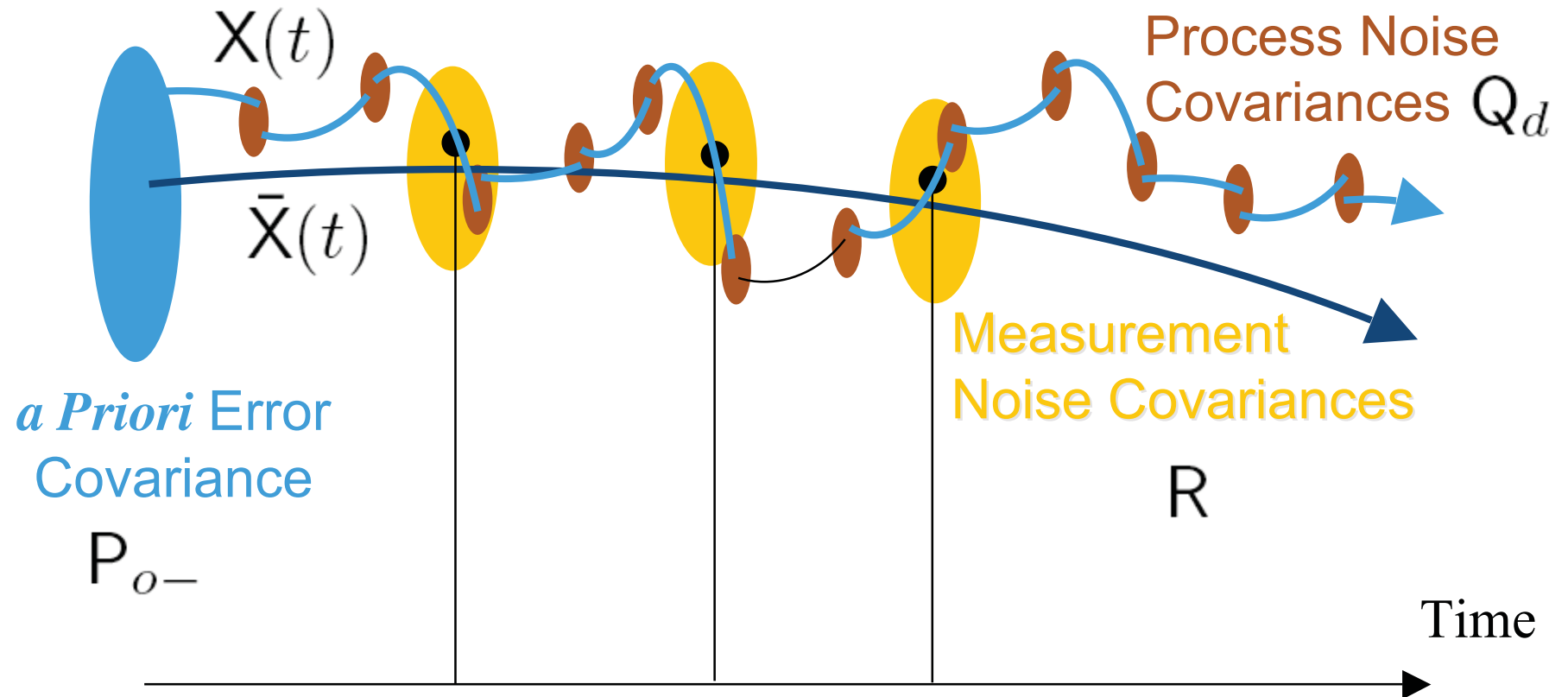
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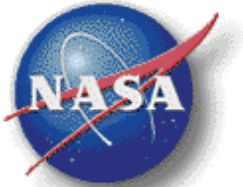




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## Context, Continued





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## Solve-for and Consider State Mapping

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### Solve-For States

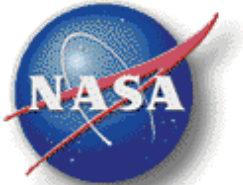
### “Consider” States

$$s(t) = S(t)x(t), \quad c(t) = C(t)x(t)$$

### Inverse Mapping:

$$M(t) = \begin{bmatrix} S(t) \\ C(t) \end{bmatrix}, \quad M^{-1}(t) = [\tilde{S}(t), \tilde{C}(t)]$$

$$x(t) = \tilde{S}(t)s(t) + \tilde{C}(t)c(t)$$



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## Covariance Partitions

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$$P = P_a + P_v + P_w, \quad \hat{P} = \hat{P}_a + \hat{P}_v + \hat{P}_w$$

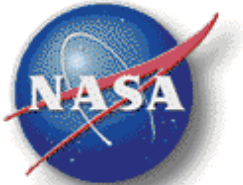
**Effect of *a Priori* Errors:**  $\Delta P_a = SP_a S^T - \hat{P}_a$

**Effect of Measurement Noise:**  $\Delta P_v = SP_v S^T - \hat{P}_v$

**Effect of Process Noise:**  $\Delta P_w = SP_w S^T - P_w$

***N.B.:  $\Delta$ 's are not necessarily positive or negative***





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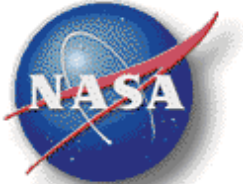
## Auxiliary Notation

### Solve-For Mapping of Dynamics and Measurements Partial

$$\Phi_{ss}(t_i, t_o) = S(t_i)\Phi(t_i, t_o)\tilde{S}(t_o), \quad H_s(t_i) = H(t_i)\tilde{S}(t_i)$$

### Process Noise Covariance Matrices

$$\hat{Q}_d(t_i, t_{i-1}) = \int_{t_{i-1}}^{t_i} \Phi_{ss}(t_i, \tau) \hat{Q}(\tau) \Phi_{ss}^T(t_i, \tau) d\tau$$
$$Q_d(t_i, t_{i-1}) = \int_{t_{i-1}}^{t_i} \Phi(t_i, \tau) Q(\tau) \Phi^T(t_i, \tau) d\tau$$



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# Sequential Filter Propagation

Only the “*a* Partition” receives the *a Priori* Errors ...

$$\hat{P}_a(t_i^-) = \Phi_{ss}(t_i, t_{i-1}) \hat{P}_a(t_{i-1}^+) \Phi_{ss}^T(t_i, t_{i-1}),$$

$$P_a(t_i^-) = \Phi(t_i, t_{i-1}) P_a(t_{i-1}^+) \Phi^T(t_i, t_{i-1}),$$

$$\hat{P}_v(t_i^-) = \Phi_{ss}(t_i, t_{i-1}) \hat{P}_v(t_{i-1}^+) \Phi_{ss}^T(t_i, t_{i-1}),$$

$$P_v(t_i^-) = \Phi(t_i, t_{i-1}) P_v(t_{i-1}^+) \Phi^T(t_i, t_{i-1}),$$

$$\hat{P}_w(t_i^-) = \Phi_{ss}(t_i, t_{i-1}) \hat{P}_w(t_{i-1}^+) \Phi_{ss}^T(t_i, t_{i-1}) + \hat{Q}_d(t_i, t_{i-1}),$$

$$P_w(t_i^-) = \Phi(t_i, t_{i-1}) P_w(t_{i-1}^+) \Phi^T(t_i, t_{i-1}) + Q_d(t_i, t_{i-1}),$$

$$\hat{P}_a(t_1^-) = \hat{P}_{o-}$$

$$P_a(t_1^-) = P_{o-}$$

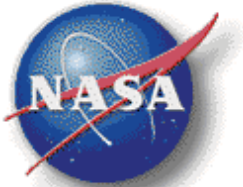
$$\hat{P}_v(t_1^-) = 0$$

$$P_v(t_1^-) = 0$$

$$\hat{P}_w(t_1^-) = 0$$

$$P_w(t_1^-) = 0$$

... and only the “*w* Partition” receives the Process Noise



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## Sequential Filter Update

**The Gain is computed from the Formal Solve-for Covariance:**

$$K_i = \hat{P}(t_i^-) H_s^T(t_i) [H_s(t_i) \hat{P}(t_i^-) H_s^T(t_i) + \hat{R}(t_i)]^{-1}$$

$$\hat{P}_a(t_i^+) = [I - K_i H_s(t_i)] \hat{P}_a(t_i^-) [I - K_i H_s(t_i)]^T$$

$$P_a(t_i^+) = [I - \tilde{S}(t_i) K_i H(t_i)] P_a(t_i^-) [I - \tilde{S}(t_i) K_i H(t_i)]^T$$

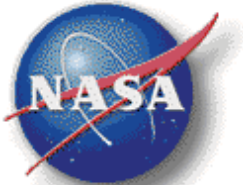
$$\hat{P}_v(t_i^+) = [I - K_i H_s(t_i)] \hat{P}_v(t_i^-) [I - K_i H_s(t_i)]^T + K_i \hat{R}(t_i) K_i^T$$

$$P_v(t_i^+) = [I - \tilde{S}(t_i) K_i H(t_i)] P_v(t_i^-) [I - \tilde{S}(t_i) K_i H(t_i)]^T + \tilde{S}(t_i) K_i R(t_i) K_i^T \tilde{S}^T(t_i)$$

$$\hat{P}_w(t_i^+) = [I - K_i H_s(t_i)] \hat{P}_w(t_i^-) [I - K_i H_s(t_i)]^T$$

$$P_w(t_i^+) = [I - \tilde{S}(t_i) K_i H(t_i)] P_w(t_i^-) [I - \tilde{S}(t_i) K_i H(t_i)]^T$$

**Only the “v Partition” receives the Measurement Noise**



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## Batch Estimator without Process Noise

We write the batch update in a form resembling that of the sequential filter to isolate the contributions of the *a priori* error and the measurement noise:

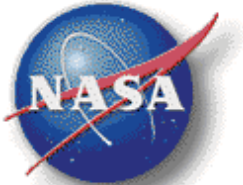
$$K_i = [\hat{P}_{*-}^{-1} + \sum_j \Phi_{ss}^T(t_j, t_*) H_s^T(t_j) \hat{R}_j^{-1} H_s(t_j) \Phi_{ss}(t_j, t_*)]^{-1} \Phi_{ss}^T(t_i, t_*) H_s^T(t_i) \hat{R}_i^{-1}$$

$$\hat{P}_a(t_*^+) = [I - \sum_i K_i H_s(t_i) \Phi_{ss}(t_i, t_*)] \hat{P}_{*-} [I - \sum_i K_i H_s(t_i) \Phi_{ss}(t_i, t_*)]^T$$

$$P_a(t_*^+) = [I - \sum_i \tilde{S}(t_i) K_i H(t_i) \Phi(t_i, t_*)] P_{*-} [I - \sum_i \tilde{S}(t_i) K_i H(t_i) \Phi(t_i, t_*)]^T$$

$$\hat{P}_v(t_*^+) = \sum_i K_i \hat{R}(t_i) K_i^T, \quad P_v(t_*^+) = \sum_i \tilde{S}(t_i) K_i R(t_i) K_i^T \tilde{S}^T(t_i)$$

We associate the *a priori* error with an anchor time that is not required to precede the measurement batch.



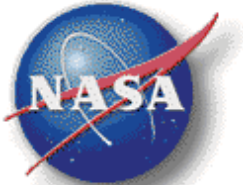
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## Effect of Process Noise on Batch Estimator

**The batch estimator does not model process noise, but if it is present, it will induce an error:**

$$P_w(t_*^+) = \Upsilon \begin{bmatrix} Q_d(t_*; t_1, t_1) & Q_d(t_*; t_1, t_2) & \cdots \\ Q_d(t_*; t_2, t_1) & Q_d(t_*; t_2, t_2) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \Upsilon^T$$

$$\Upsilon = \begin{bmatrix} \tilde{S}(t_1)K_1H(t_1)\Phi(t_1, t_*) & \tilde{S}(t_2)K_2H(t_2)\Phi(t_2, t_*) & \cdots \end{bmatrix}$$



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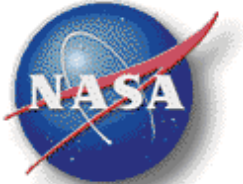
## Generalized Process Noise Covariance

**Process noise enters the innovations, and thus must be mapped to the anchor time:**

$$Q_d(t_*; t_i, t_j) = \begin{cases} \int_{t_*}^{\min(t_i, t_j)} \Phi(t_i, \tau) Q(\tau) \Phi^T(t_j, \tau) d\tau & t_* < t_i, t_* < t_j, \\ \int_{\max(t_i, t_j)}^{t_*} \Phi(t_i, \tau) Q(\tau) \Phi^T(t_j, \tau) d\tau & t_i < t_*, t_j < t_*, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} Q_d(t_i, t_*) \Phi^T(t_j, t_i) & t_* < t_i \leq t_j, \\ \Phi(t_i, t_j) Q_d(t_j, t_*) & t_* \leq t_j \leq t_i, \\ \Phi(t_i, t_j) Q_d(t_*, t_j) & t_i \leq t_j < t_*, \\ Q_d(t_*, t_i) \Phi^T(t_j, t_i) & t_j \leq t_i < t_*, \\ 0 & \text{otherwise} \end{cases}$$

**If the anchor time is between a pair of measurements, their process noise contributions do not overlap, so the expected contribution to the covariance is zero**



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## Sensitivity to Individual *a Priori* Parameters

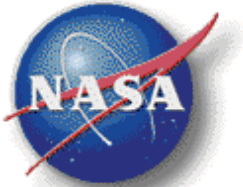
- In this work, a “sensitivity” is a matrix of partial derivatives of some parameters of interest with respect to others (cf. Gelb, where “sensitivity” refers to a covariance matrix)
- The “ $\Delta P$ ’s” give an indication of sensitivities of solve-for covariances to groups of errors (*a priori*, measurement noise, and process noise)
- Often we’d also like to know the sensitivity of each solve-for at any point in time to each individual *a priori* solve-for or consider error

### Sensitivity of solve-fors with respect to *a priori*’s (sequential form):

$$\Sigma_a(t_i) = [I - \tilde{S}(t_i)K_i H(t_i)]\Phi(t_i, t_{i-1})\Sigma_a(t_{i-1}), \quad \Sigma_a(t_o) = [\tilde{S}(t_o), \tilde{C}(t_o)]$$

### Sensitivity of solve-fors with respect to *a priori*’s (batch form):

$$\Sigma_a(t) = S(t)\Phi(t, t_*)[I - \sum_{i=1}^k \tilde{S}(t_i)K_i H(t_i)\Phi(t_i, t_*)][\tilde{S}(t_*), \tilde{C}(t_*)]$$



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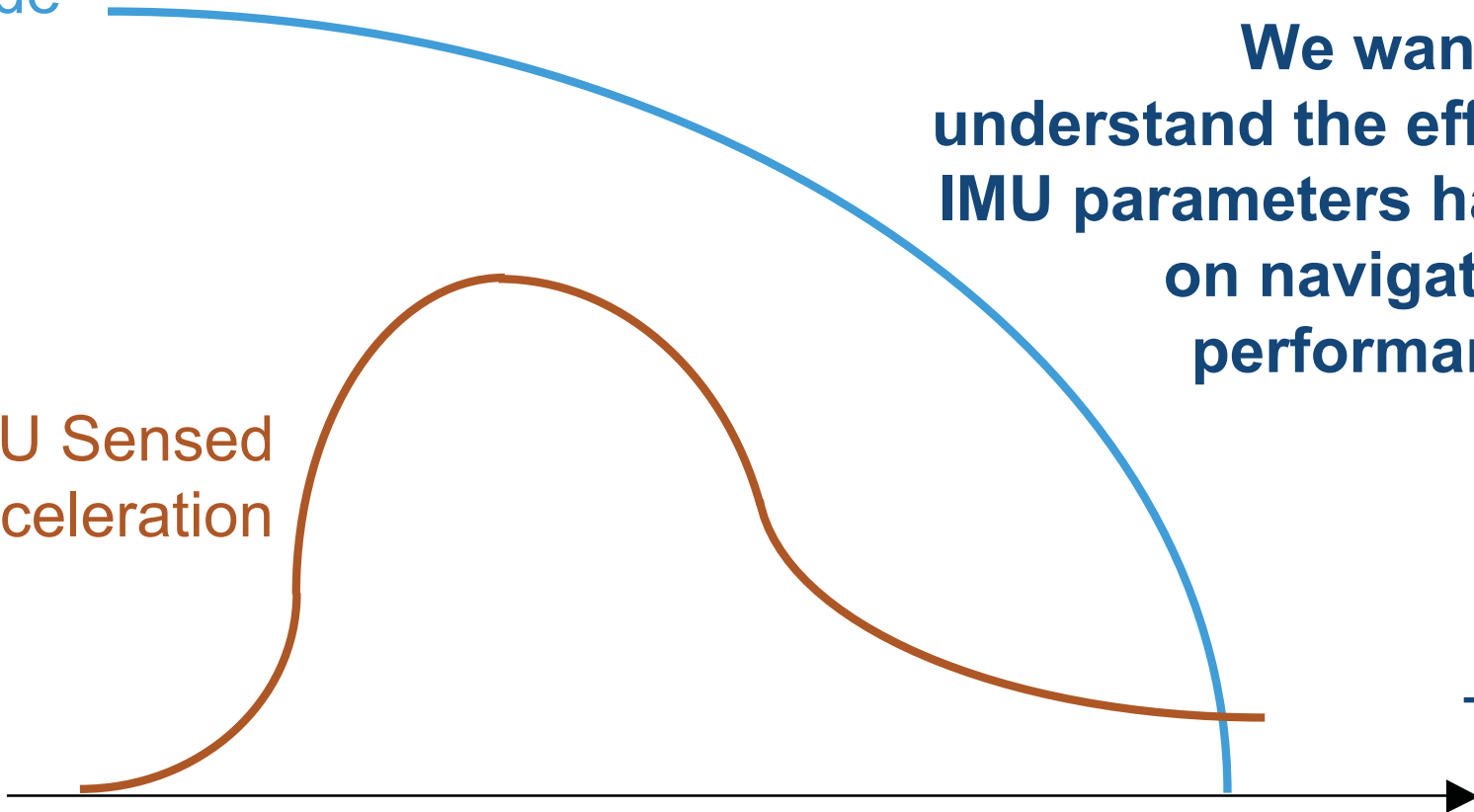
## Example: Entry Problem

Altitude

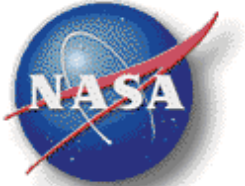
IMU Sensed  
Acceleration

**We want to  
understand the effect  
IMU parameters have  
on navigation  
performance**

Time







# IMU Model

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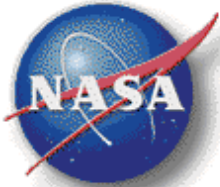
$\mathbf{x}_g = [\boldsymbol{\theta}_C^T, \mathbf{b}_{gC}^T, \mathbf{s}_g^T, \boldsymbol{\gamma}_g^T]^T$  **Gyro state: angular misalignment, rate bias, scale factor, non-orthogonality**

$$\begin{pmatrix} \frac{c_d}{dt} \boldsymbol{\theta}_C \\ \frac{c_d}{dt} \mathbf{b}_{gC} \\ \frac{d}{dt} \mathbf{s}_g \\ \frac{d}{dt} \boldsymbol{\gamma}_g \end{pmatrix} = \begin{bmatrix} \mathbf{O}_{3 \times 3} & \mathbf{I}_3 & \mathbf{D}(\mathcal{I} \boldsymbol{\omega}_C^C) & \mathbf{F}(\mathcal{I} \boldsymbol{\omega}_C^C) \\ & & \mathbf{O}_{9 \times 12} & \end{bmatrix} \mathbf{x}_g + \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{O}_{9 \times 3} \end{bmatrix} \boldsymbol{\epsilon}_\omega$$

$$\dot{\mathbf{x}}_g = \mathbf{A}_g \mathbf{x}_g + \mathbf{B}_g \boldsymbol{\epsilon}_\omega,$$

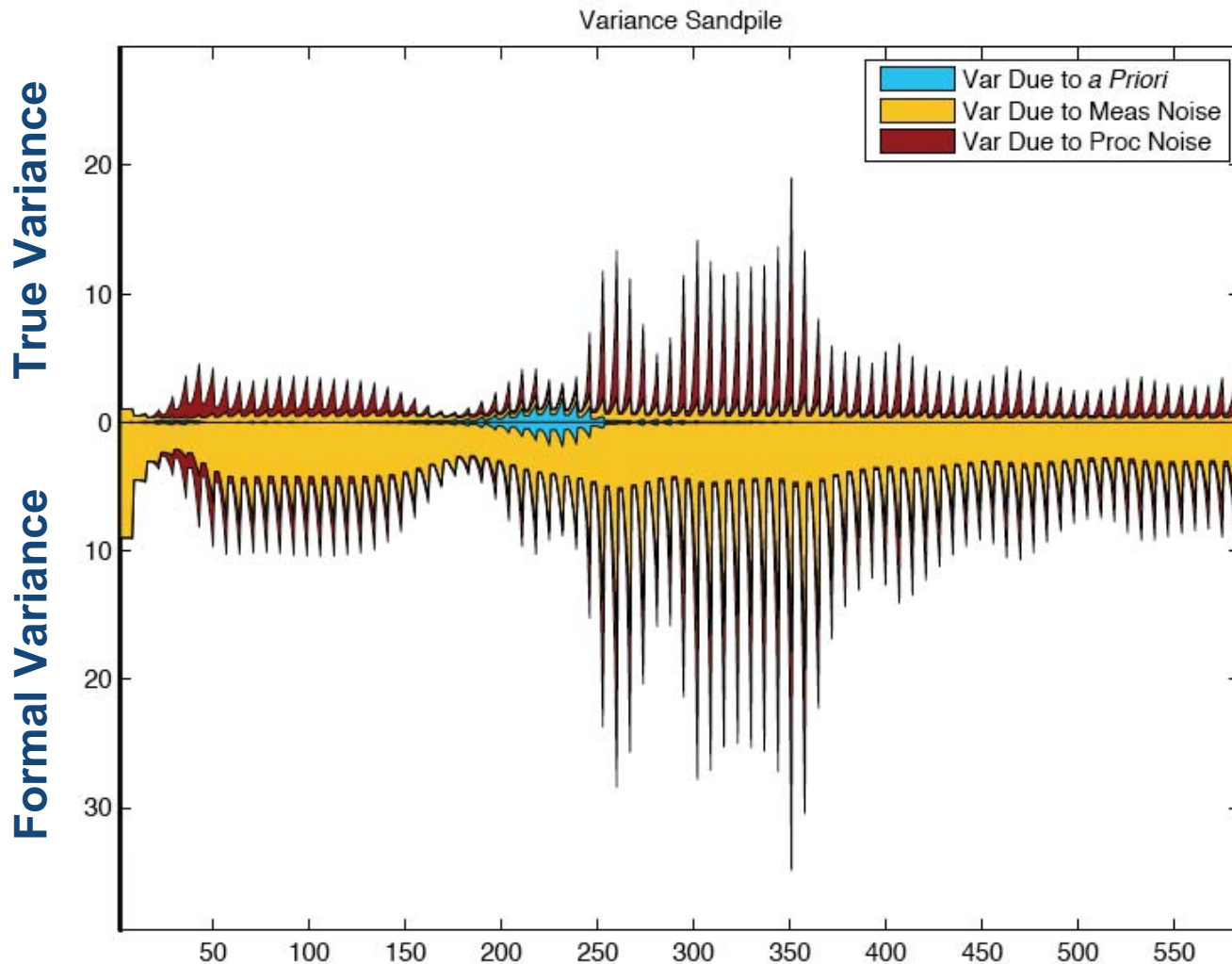
$\mathbf{x}_a = [\mathbf{b}_{aC}^T, \mathbf{s}_a^T, \boldsymbol{\gamma}_a^T]^T$  **Accelerometer state: acceleration bias, scale factor, non-orthogonality**

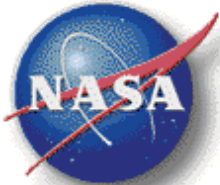
$$\begin{aligned} \delta \mathbf{a}_C &= \begin{bmatrix} \mathbf{I}_3 & \mathbf{D}(\mathbf{a}_C) & \mathbf{F}(\mathbf{a}_C) \end{bmatrix} \mathbf{x}_a \\ &= \mathbf{H}_a \mathbf{x}_a. \end{aligned}$$



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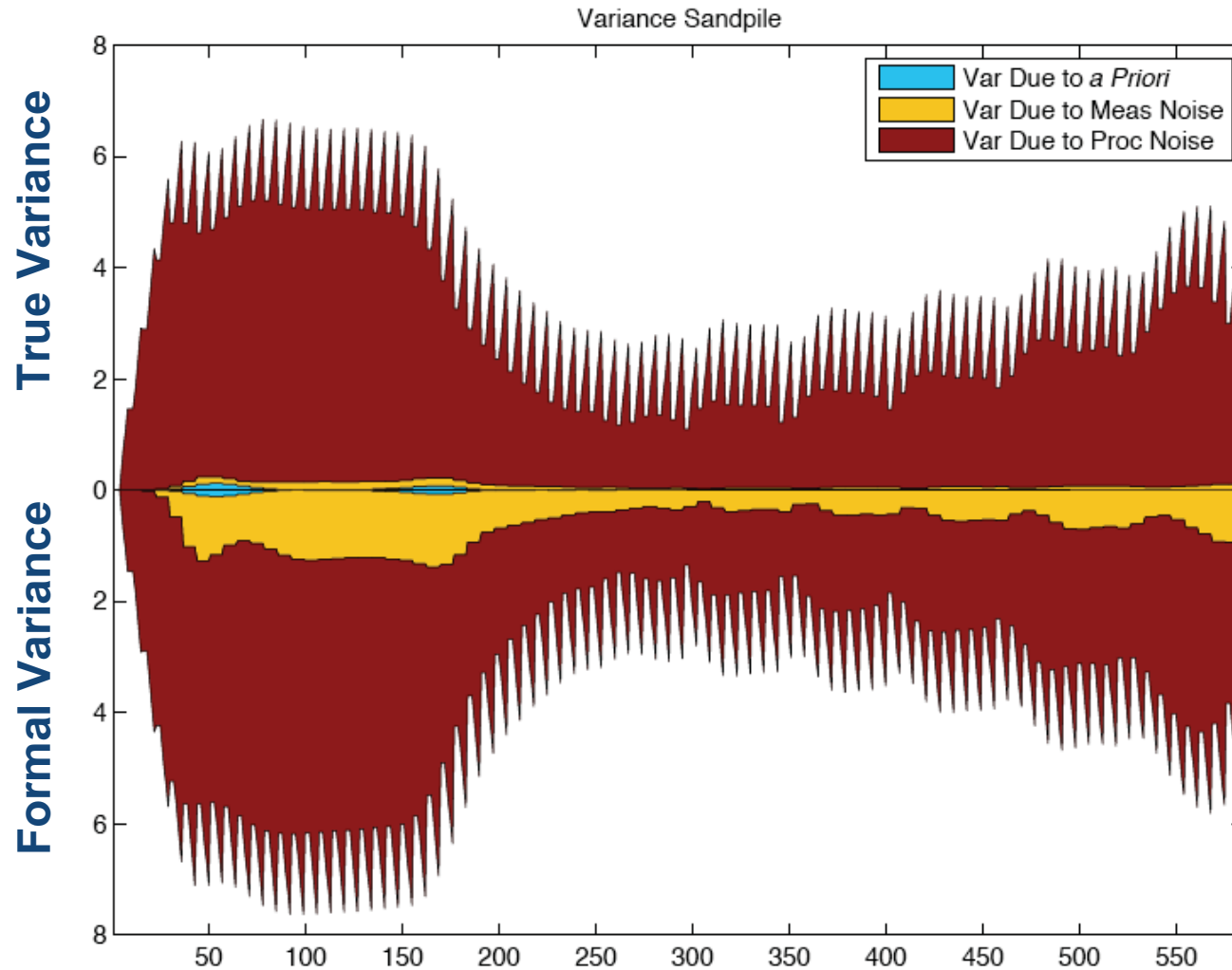
# Variance Sandpile: *Y*-Component of Inertial Position Error

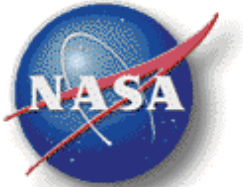




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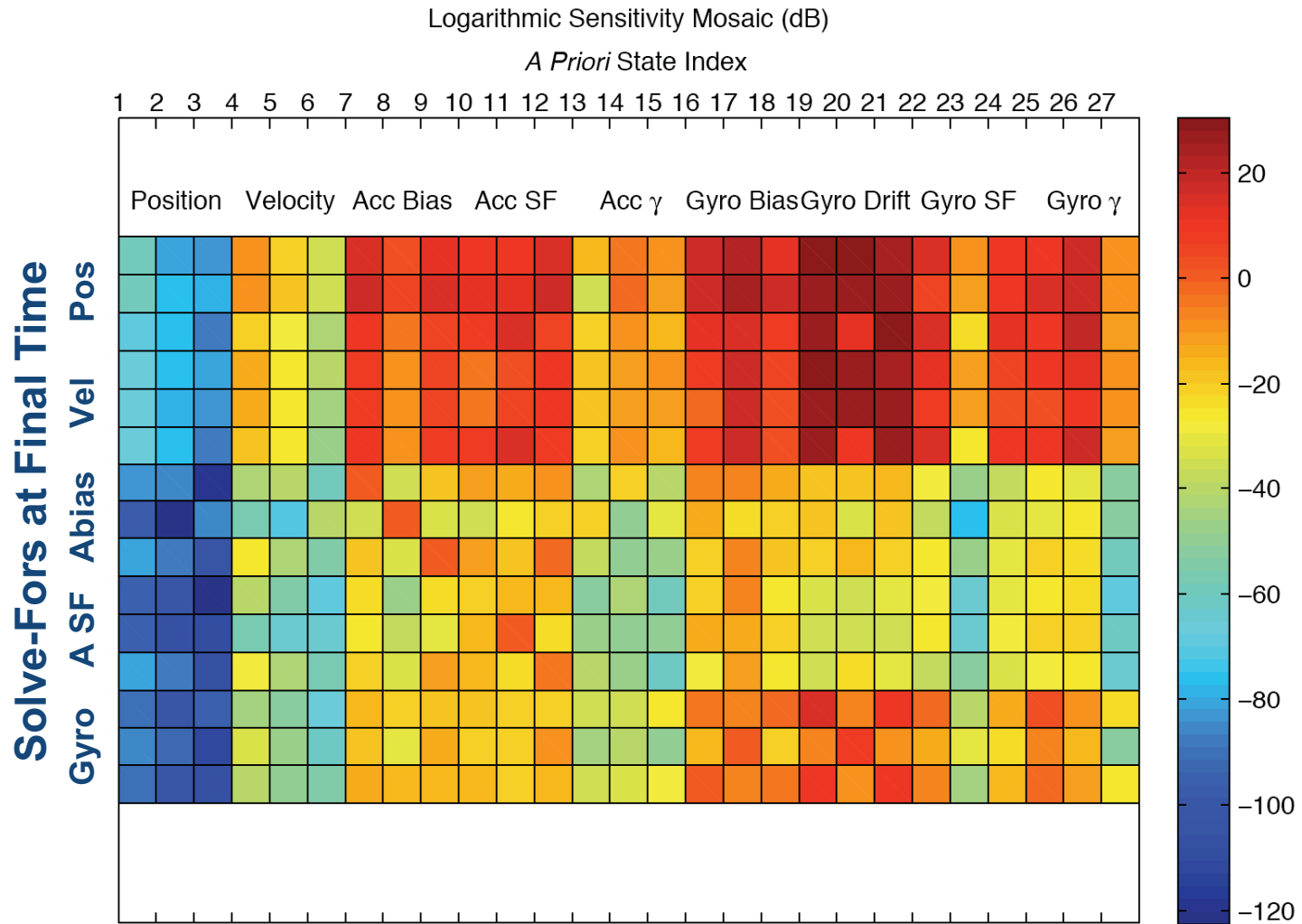
## Variance Sandpile: $Y$ -Component of Case-Fixed Gyro Angular Error

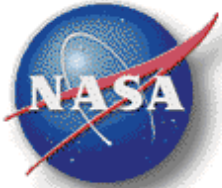




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# Sensitivity Mosaic





# Summary

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- Present work updates to Markley et al.
- Augments approach to sensitivity analysis
- Addresses “postdiction” in the batch framework
- Applies method to integrated orbit/attitude problem
- Includes some new ways to examine output

$$\Delta P_a = SP_a S^T - \hat{P}_a$$

$$\Delta P_v = SP_v S^T - \hat{P}_v$$

$$\Delta P_w = SP_w S^T - P_w$$

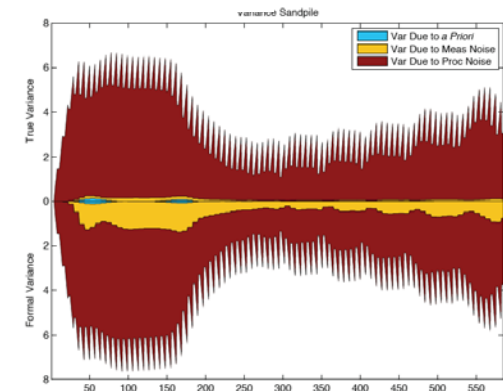
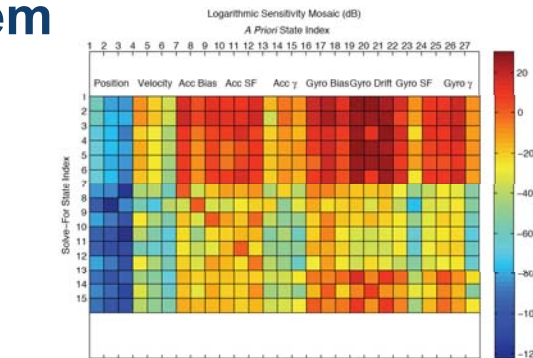
$$M(t) = \begin{bmatrix} S(t) \\ C(t) \end{bmatrix}, \quad M^{-1}(t) = [\tilde{S}(t), \tilde{C}(t)]$$

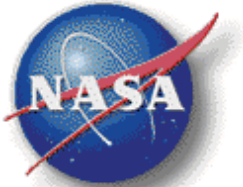
$$x(t) = \tilde{S}(t)s(t) + \tilde{C}(t)c(t)$$

$$Q_d(t_*; t_i, t_j) = \begin{cases} \int_{t_*}^{\min(t_i, t_j)} \Phi(t_i, \tau) Q(\tau) \Phi^T(t_j, \tau) d\tau & t_* < t_i, t_* < t_j, \\ \int_{\max(t_i, t_j)}^{t_*} \Phi(t_i, \tau) Q(\tau) \Phi^T(t_j, \tau) d\tau & t_i < t_*, t_j < t_*, \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{pmatrix} \frac{x_d}{dt} \delta \mathbf{r}_I \\ \frac{x_d}{dt} \delta \mathbf{v}_I \\ \dot{x}_a \\ \dot{x}_g \end{pmatrix} = \begin{bmatrix} \mathbf{O}_{3 \times 3} & \mathbf{I}_3 & \mathbf{O}_{3 \times 9} & \mathbf{O}_{3 \times 12} \\ \mathbf{G}(\mathbf{r}_I) & \mathbf{O}_{3 \times 3} & \mathbf{M}_C^T \mathbf{H}_a & \mathbf{M}_C^T \mathbf{X}_{ag} \\ \mathbf{O}_{12 \times 3} & \mathbf{O}_{12 \times 3} & \mathbf{O}_{9 \times 9} & \mathbf{O}_{12 \times 12} \\ \mathbf{O}_{12 \times 3} & \mathbf{O}_{12 \times 3} & \mathbf{O}_{9 \times 9} & \mathbf{A}_g \end{bmatrix} \begin{pmatrix} \delta \mathbf{r}_I \\ \delta \mathbf{v}_I \\ x_a \\ x_g \end{pmatrix} + \begin{bmatrix} \mathbf{O}_{15 \times 3} \\ \mathbf{B}_g \end{bmatrix} \epsilon_\omega$$

$$\dot{x} = Ax + B\epsilon_\omega$$





# INS Model

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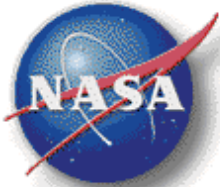
**IMU sensed acceleration error integrates into position and velocity errors according to the INS model:**

$$\begin{pmatrix} \frac{d}{dt} \delta \mathbf{r}_{\mathcal{I}} \\ \frac{d}{dt} \delta \mathbf{v}_{\mathcal{I}} \\ \dot{x}_a \\ \dot{x}_g \end{pmatrix} = \begin{bmatrix} \mathbf{O}_{3 \times 3} & \mathbf{I}_3 & \mathbf{O}_{3 \times 9} & \mathbf{O}_{3 \times 12} \\ \mathbf{G}(\mathbf{r}_{\mathcal{I}}) & \mathbf{O}_{3 \times 3} & \mathbf{M}_{\mathcal{C}}^{\mathcal{I}} \mathbf{H}_a & \mathbf{M}_{\mathcal{C}}^{\mathcal{I}} \mathbf{X}_{ag} \\ \mathbf{O}_{12 \times 3} & \mathbf{O}_{12 \times 3} & \mathbf{O}_{9 \times 9} & \mathbf{O}_{12 \times 12} \\ \mathbf{O}_{12 \times 3} & \mathbf{O}_{12 \times 3} & \mathbf{O}_{9 \times 9} & \mathbf{A}_g \end{bmatrix} \begin{pmatrix} \delta \mathbf{r}_{\mathcal{I}} \\ \delta \mathbf{v}_{\mathcal{I}} \\ x_a \\ x_g \end{pmatrix} + \begin{bmatrix} \mathbf{O}_{15 \times 3} \\ \mathbf{B}_g \end{bmatrix} \boldsymbol{\varepsilon}_{\omega}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\varepsilon}_{\omega}$$

**where**

$$\mathbf{X}_{ag} = [\mathbf{a}_{\mathcal{C}}^{\times}, \mathbf{O}_{3 \times 9}]$$



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## Example Problem Parameters

| Simulation Parameter              | Value               | Units                      |
|-----------------------------------|---------------------|----------------------------|
| Gravitational Constant            | $4.305 \times 10^4$ | $\text{km}^3/\text{sec}^2$ |
| Measurement Time Interval         | 2                   | sec                        |
| Estimation Parameter              | Standard Deviation  | Units                      |
| True Position Measurement Noise   | 0.305               | m                          |
| Formal Position Measurement Noise | 0.914               | m                          |
| Initial Position Error            | 30.5                | meters                     |
| Initial Velocity Error            | 3.05                | cm/sec                     |
| Accelerometer Bias                | 60                  | $\mu g$                    |
| Accelerometer Scale Factor        | 500                 | ppm                        |
| Accelerometer Nonorthogonality    | 10                  | ppm                        |
| Initial Gyro Angular Error        | 42                  | arcsec                     |
| Gyro Bias Drift                   | 0.01                | deg/hr                     |
| Gyro Scale Factor                 | 33                  | ppm                        |
| Gyro Nonorthogonality             | 20                  | ppm                        |
| Gyro Random Walk Intensity        | 0.025               | $\text{deg/hr}^{1/2}$      |